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CLASSICAL BEHAVIOR OF A MACROSCOPIC SCHRÖDINGER CAT

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Abstract

We study the dynamics of classical and quantum systems linearly interacting with a classical environment represented by an infinite set of harmonic oscillators. The environment induces a dynamical localization of the quantum state into a generalized coherent state for which the $\hbar \to 0$ limit always exists and reproduces the classical motion. We describe the consequences of this localization on the behavior of a macroscopic system by considering the example of a Schrödinger cat.

INTRODUCTION

The problem of how classical behavior is regained from quantum mechanics in the macroscopic limit can be conceptually solved by recognizing that a macroscopic system is never completely isolated by the external world. It has been argued[1, 2, 3] that the interaction with an environment can, after a transient whose duration presumably depends on the coupling strength, drive the totality of the admissible states of the Hilbert space into those having classical limit, formally $\hbar \to 0$.

In a recent paper in collaboration with R. Onofrio and M. Patriarca[4], we substantiated this conjecture by analyzing the dynamics of general classical and quantum systems linearly interacting with an infinite set of degrees of freedom. Here, we briefly review the main results of this model and describe in detail how the pathologies of a simple Schrödinger cat are cured by the presence of the environment.

Let us consider a system described by the classical Hamiltonian

$$H(p,q,t) = \frac{p^2}{2m} + V(q,t) \tag{1}$$

We model its interaction with a classical environment by a linear coupling to an infinite set of degrees of freedom $\{P_n, Q_n\}$. The Hamiltonian for the total system is

$$H_{tot} = H(p,q,t) + H_m(P, Q-q), \tag{2}$$

where

$$H_m(P,Q-q) = \sum_n \left[\frac{P_n^2}{2M} + \frac{M\omega_n^2}{2}(Q_n - q)^2\right]. \tag{3}$$

The classical dynamics of the system modified by the environment is described in terms of equations obtained by formally solving the harmonic motion of $\{P_n, Q_n\}$. For an environment having frequencies $\{\omega_n\}$ distributed with density $dN/d\omega = \theta(\Omega - \omega)2m\gamma/\pi M\omega^2$, we get, for times $\gg \Omega^{-1}$, the Markovian evolution

$$dp(t) = -\left[\gamma p(t) + \partial_q V(q(t),t)\right]dt + \sqrt{2m\gamma k_B T}\xi(t)dt \tag{4}$$

$$dq(t) = \frac{p(t)}{m}dt \tag{5}$$

with initial conditions p' and q' at time t'. If the the initial conditions $\{P_n', Q_n'\}$ of the environment are chosen as a realization of the equilibrium Gibbs measure at temperature T, then $\xi(t)$ is a realization of a stochastic process in time with respect to the same measure with properties $\overline{\xi(t)} = 0$ and $\overline{\xi(t)\xi(s)} = \delta(t-s)$. Therefore, Eqs. (4,5) are stochastic Langevin equations.

In alternative to the detailed stochastic description, we may be interested to determine the average behavior of the system obtained by considering all the possible realizations of the initial conditions of the environment. In this case, the system is described by a probability density $W(p,q,t)$ solution of the Fokker-Plank equation associated to (4-5)

$$\partial_t W(p,q,t) = \left[-\frac{p}{m}\partial_q + \partial_q V(q,t)\partial_p + \partial_p\left(\gamma p + m\gamma k_B T \partial_p\right)\right] W(p,q,t) \tag{6}$$

with initial conditions $W(p,q,t') = \delta(p-p')\delta(q-q')$.

The classical analysis can be repeated at quantum level. Besides obvious technical modifications, there is now a conceptual difference. Since we do not know how to describe the coupling of classical and quantum degrees of freedom, we must start with a quantum description of both the system and the environment. The condition of classical behavior of the environment can be reintroduced later by asking that the thermal energy $k_B T$ is much larger than the energy spacing of the highest-frequency oscillators $\hbar\Omega$. If this high temperature condition is satisfied, for times $\gg \Omega^{-1}$ the system is described by the nonlinear stochastic Schrödinger equation

$$\begin{aligned} d|\psi_{[\xi]}(t)\rangle = & -\frac{i}{\hbar}\left[\hat{H}(\hat{p},\hat{q},t) + \frac{\gamma}{4}(\hat{p}\hat{q} + \hat{q}\hat{p})\right]|\psi_{[\xi]}(t)\rangle dt \\ & -\frac{1}{2}\left[\hat{A}^\dagger\hat{A} + a(t)^* a(t) - 2a(t)^*\hat{A}\right]|\psi_{[\xi]}(t)\rangle dt + \left[\hat{A} - a(t)\right]|\psi_{[\xi]}(t)\rangle\xi(t)dt, \end{aligned} \tag{7}$$

where $\hat{A} = \sqrt{2m\gamma k_B T/\hbar^2}\,\hat{q} + i\sqrt{\gamma/8mk_B T}\,\hat{p}$, $a(t) = \langle\psi_{[\xi]}(t)|\hat{A}|\psi_{[\xi]}(t)\rangle$, and $\xi(t)$ is a real white noise.

A direct characterization of the average properties of the quantum system is also possible. By introducing the reduced density matrix operator

$$\hat{\rho}(t) = \overline{|\psi_{[\xi]}(t)\rangle\langle\psi_{[\xi]}(t)|} \tag{8}$$

and the associated Wigner function

$$W(p,q,t) = \frac{1}{2\pi\hbar}\int dz\ \exp\left(\frac{i}{\hbar}pz\right)\langle q-\frac{z}{2}|\hat{\varrho}(t)|q+\frac{z}{2}\rangle, \tag{9}$$

from Eq. (7) we obtain

$$\begin{aligned}\partial_t W(p,q,t) = \Big[&-\frac{p}{m}\partial_q + \sum_{n=0}^{\infty}\left(\frac{\hbar}{2i}\right)^{2n}\frac{1}{(2n+1)!}\partial_q^{2n+1}V(q,t)\partial_p^{2n+1} \\ &+ \partial_p\left(\gamma p + m\gamma k_B T\partial_p\right) + \frac{\hbar^2\gamma}{16mk_BT}\partial_q^2\Big]W(p,q,t). \end{aligned} \tag{10}$$

Note that (7) and (10) reduce to the corresponding quantum equations for an isolated system when $\gamma = 0$.

DYNAMICAL LOCALIZATION INTO A COHERENT STATE

In the previous section, we have described the equations which govern the dynamics of classical and quantum systems in interaction with a classical environment. Now, we show that the quantum dynamics reduces to the classical one when the limit $\hbar \to 0$ is taken. Note that this is not always possible in the case of an isolated system where well known pathological limits are encountered[5].

It is possible to demonstrate rigorously[6,4] that for a system with potential

$$V(q,t) = v_0(t) + v_1(t)q + \frac{1}{2}m\omega_0^2 q^2 \tag{11}$$

after a time which, in the worst case, is of the order of $\sqrt{\hbar/\gamma k_B T}$ the solutions of Eq. (7) become of the form

$$|\psi_{[\xi]}(t)\rangle = \exp\left[-\frac{i}{\hbar}\varphi(t)\right]\ |p(t)q(t)\rangle, \tag{12}$$

where $\varphi(t)$ is real, $p(t) = \langle\psi_{[\xi]}(t)|\hat{p}|\psi_{[\xi]}(t)\rangle$, $q(t) = \langle\psi_{[\xi]}(t)|\hat{q}|\psi_{[\xi]}(t)\rangle$, and

$$\langle q|p(t)q(t)\rangle = (2\pi\sigma_q^2)^{-1/4}\exp\left\{-\frac{1-\frac{2i}{\hbar}\sigma_{pq}^2}{4\sigma_q^2}\ [q-q(t)]^2 + \frac{i}{\hbar}p(t)[q-q(t)]\right\}. \tag{13}$$

The state (13) defined in terms of the parameters

$$\sigma_q^2 = \frac{\hbar}{m}\sqrt{\frac{\gamma^2-\omega_0^2+\sqrt{(\gamma^2-\omega_0^2)^2+16\left(\gamma k_B T/\hbar\right)^2}}{32\left(\gamma k_B T/\hbar\right)^2}} \tag{14}$$

$$\sigma_{pq}^2 = \sqrt{m^2(\gamma^2-\omega_0^2)\sigma_q^4 + \frac{\hbar^2}{4}} - m\gamma\sigma_q^2 \tag{15}$$

is a generalized coherent state which admits the $\hbar \to 0$ limit. Moreover, the convergence into this coherent state takes place in a time which vanishes for $\hbar \to 0$. Therefore, a

linear system like (11) always has classical limit at any time $t > t'$ even if the $\hbar \to 0$ limit does not exist at the initial time t'. Since the contribution to the Green function of Eq. (7) due the system-environment coupling is of the form $q^2\hbar^{-1}$, these results apply in the limit $\hbar \to 0$ also to nonlinear systems.

We illustrate the consequences of the localization into a coherent state induced by the coupling with the environment by analyzing the evolution of a cat state. For simplicity, consider a free quantum particle which at time $t' = 0$ is in the superposition state

$$|\psi(0)\rangle = N\left(|\frac{1}{2}P - \frac{1}{2}Q\rangle + | - \frac{1}{2}P\frac{1}{2}Q\rangle\right) \tag{16}$$

where $| \pm \frac{1}{2}P \pm \frac{1}{2}Q\rangle$ are coherent states (13-15) with $\omega_0 = 0$ and N is a normalization factor. The state (16) has no classical counterpart. Indeed, its corresponding Wigner function

$$W(p,q,0) = N^2 \left\{ W_{\frac{1}{2}P-\frac{1}{2}Q}(p,q) + W_{-\frac{1}{2}P\frac{1}{2}Q}(p,q) + W_{00}(p,q)\ 2\cos\left[\frac{p}{\hbar}Q + \frac{q}{\hbar}P\right]\right\} \tag{17}$$

does not have $\hbar \to 0$ limit due to the presence of the last oscillating term. In Eq. (17), we indicated with $W_{p_0q_0}(p,q)$ the Wigner function of the coherent state $|p_0q_0\rangle$ having limit $W_{p_0q_0}(p,q) \xrightarrow{\hbar\to 0} \delta(p-p_0)\delta(q-q_0)$. In more physical terms, if we consider a macroscopic limit, for instance increase the values of P and/or Q, the oscillating interference term in (17) never disappear contrarily to common sense. An example of this pathological behavior is shown in Fig. 1.

At a later time, the situation is different. By solving Eq. (10) with the initial condition (17), we find the following expression for the Wigner function

$$\begin{aligned} W(p,q,t) = N^2 \Big\{ & W_{\frac{1}{2}P-\frac{1}{2}Q}(p,q,t) + W_{-\frac{1}{2}P+\frac{1}{2}Q}(p,q,t) + W_{00}(p,q,t) \\ & \times e^{-C_{P-Q}+\Sigma_{P-Q}(t)}\ 2\cos\left[p\Upsilon_{P-Q}(t) + q\Phi_{P-Q}(t)\right]\Big\}. \end{aligned} \tag{18}$$

The definitions of $W_{p_0q_0}(p,q,t)$, C_{P-Q}, $\Sigma_{P-Q}(t)$, $\Upsilon_{P-Q}(t)$, and $\Phi_{P-Q}(t)$ can be found in Ref. 4. The functions $W_{p_0q_0}(p,q,t)$ are solution of Eq. (10) with initial condition $W_{p_0q_0}(p,q)$. In the $\hbar \to 0$ limit, they reduce to phase-space probability densities $W^{\mathrm{cl}}_{p_0q_0}(p,q,t)$ solution of the classical Fokker-Plank equation (6) with initial condition $\delta(p-p_0)\delta(q-q_0)$. The exponential term $\exp[-C_{P-Q} + \Sigma_{P-Q}(t)]$ vanishes for $\hbar \to 0$ at any $t > 0$ and

$$\lim_{\hbar\to 0} W(p,q,t) = \frac{1}{2}\left[W^{\mathrm{cl}}_{\frac{1}{2}P-\frac{1}{2}Q}(p,q,t) + W^{\mathrm{cl}}_{-\frac{1}{2}P\frac{1}{2}Q}(p,q,t)\right]. \tag{19}$$

The classical limit is equivalently reached for large values of P and/or Q. An estimate of the critical values of P and Q for which the Wigner function changes from (18) to (19) can be obtained by observing that $\Sigma_{P-Q}(t) = C_{P-Q} - t/\tau + \mathcal{O}(t^2)$ with the characteristic time τ given by

$$\gamma\tau \simeq \left(\frac{k_BT}{\hbar\gamma}\ \frac{\hbar Q^2}{m\gamma} + \frac{1}{16}\frac{\hbar\gamma}{k_BT}\ \frac{P^2}{\hbar m\gamma}\right)^{-1}. \tag{20}$$

Note that $\tau(P,Q) \to 0$ when P and/or Q diverge. At a chosen time $t > 0$, we have quantum behavior for $\tau(P,Q) \gg t$ (microscopic system) and classical behavior for $\tau(P,Q) \ll t$ (macroscopic system). An example of this quantum-to-classical transition is shown in Fig. 2.

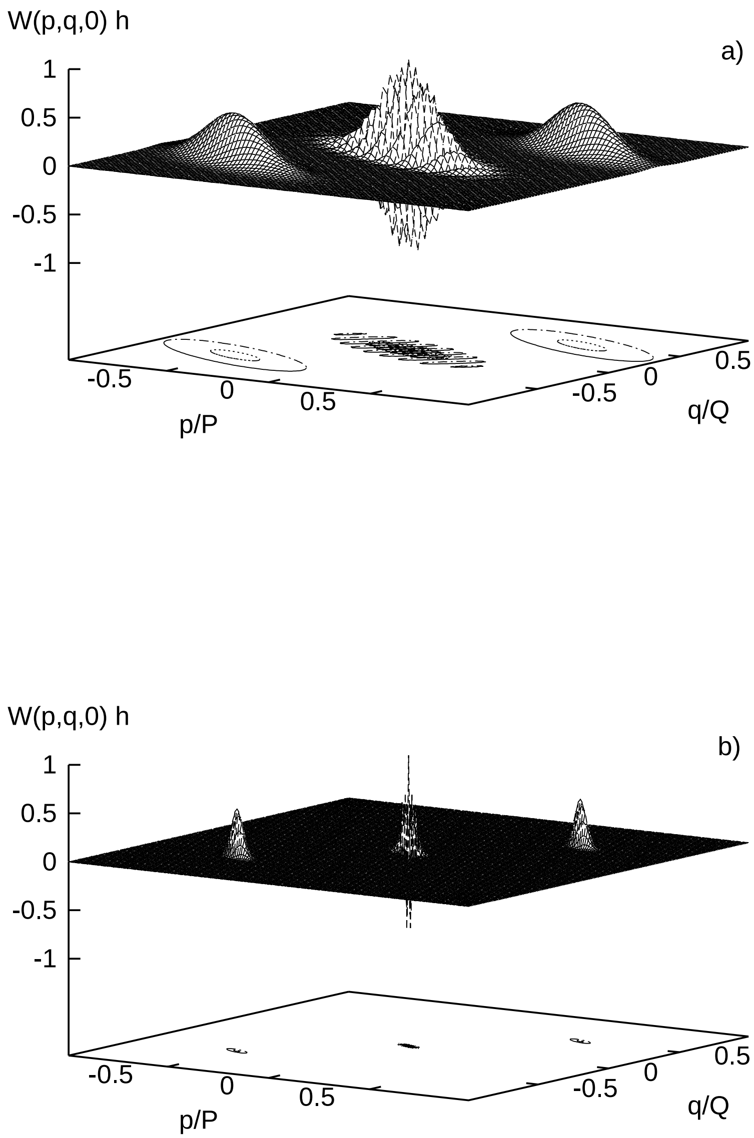

Figure 1. Wigner function at time 0 corresponding to the cat state (16) for $k_BT/\hbar\gamma = 100$. In case a) we have $P = 20\sqrt{\hbar m\gamma}$ and $Q = 2\sqrt{m\gamma/\hbar}$ while in b) $P = 100\sqrt{\hbar m\gamma}$ and $Q = 10\sqrt{m\gamma/\hbar}$.

Finally, we note that $\Sigma_{P-Q}(t)$, $\Upsilon_{P-Q}(t)$, and $\Phi_{P-Q}(t)$ vanish for $t \to \infty$ so that in this limit the Wigner function becomes

$$W_\infty(p,q,t) = N^2 \left\{ W^{\mathrm{cl}}_{\frac{1}{2}P-\frac{1}{2}Q}(p,q,t) + W^{\mathrm{cl}}_{-\frac{1}{2}P\frac{1}{2}Q}(p,q,t) + W^{\mathrm{cl}}_{00}(p,q,t)e^{-C_{P-Q}}2 \right\}. \quad (21)$$

The classical limit, formally $\hbar \to 0$, is reached only for a macroscopic system, not in the long time limit of a microscopic one.

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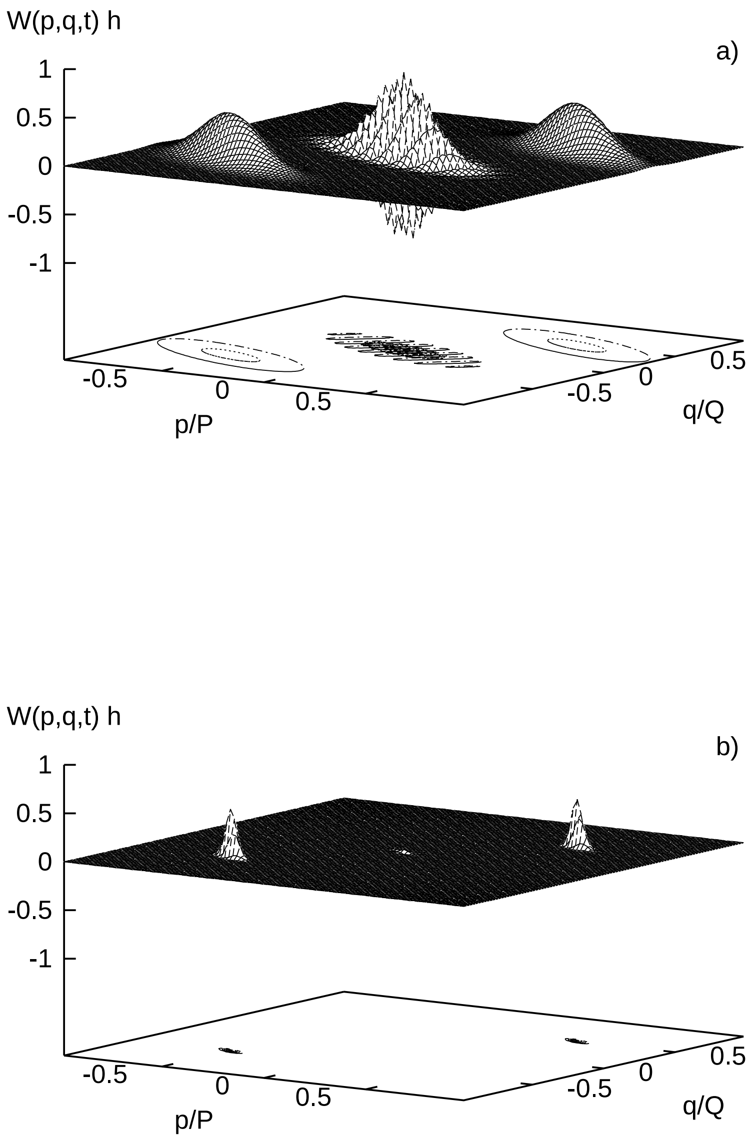

Figure 2. As in Fig. 1 but at the adimensional time $\gamma t = \frac{1}{2}10^{-3}$. Note that the adimensional characteristic time (20) is about $\frac{1}{4}10^{-2}$ in case a) and 10^{-4} in case b).

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